Three-dimensional three-wave interactions: A bilinear approach

1998 J. Phys. A: Math. Gen. 31 349

(http://iopscience.iop.org/0305-4470/31/1/030)

J. Phys. A: Math. Gen. **31** (1998) 349–367. Printed in the UK PII: S0305-4470(98)86219-7

Three-dimensional three-wave interactions: A bilinear approach

Claire R Gilson and Mark C Ratter

Department of Mathematics, University Gardens, University of Glasgow, Glasgow G12 8QW, UK

Received 23 July 1997, in final form 2 October 1997

Abstract. We consider the three-dimensional three-wave resonance equation using a bilinear approach to investigate a broad class of solutions. Solutions are obtained in a Grammian form, and their relationship to Kaup's solutions examined.

1. Introduction

Of the nonlinear interactions possible in three dimensions, possibly the simplest is the three-wave resonant interaction. If we consider two colliding waves whose envelopes vary slowly compared with their central frequencies (w_1, w_2), it is possible for a third wave to exist which has a composite frequency ($w_3 = w_1 \pm w_2$). It may be viewed as the two original waves interacting (beating) with one another, giving rise to the third wave. Given the frequency of the third is a linear combination of the first two, the waves can 'phase lock' allowing for growth and continuation of the new wave. Without this feature the third wave would quickly disperse after several oscillations. For a full discussion on the derivation of the equations and the physical significance see [5].

The three-wave resonant interaction (3WRI) has been extensively investigated. Originally this work was in one spatial dimension rather than the full three-dimensional problem. An inverse-scattering transform for the homogenous medium 3WRI in one space dimension and time was developed by Zakharov and Manakov [20, 21] and Kaup [10]. A full discussion of the evolution in time and one-spatial dimension, in a homogenous medium of the 3WRI was made by Kaup *et al* [15] and for the inhomogeous medium case by Reiman [16].

The above work led the way for a study of the full three-dimensional three-wave resonant (3D3WR) interaction, which arises as the compatability condition between two 3×3 differential systems. The inverse-scattering problem was first formulated by Zakharov [19] who found a particular class of solutions, and then Craik [4], independently, investigated in detail some special cases of these solutions. These solutions came to be known as 'lumps'. Ablowitz and Haberman's work [1] led Cornille [3] to reformulate the inverse-scattering problem with all three coordinates on the same footing and Kaup [12, 11, 14] used characteristic coordinates to give explicitly the general inverse-scattering solution and also an infinite set of conservation laws.

The lump solutions obtained by Zakharov and later Craik are different from the soliton solutions found in the one-dimensional inverse-scattering, however, they maintain some similarities. The one-dimensional soliton solutions are derived from separable kernels [15]

0305-4470/98/010349+19$19.50

and hence give rise to a closed form of solution. Similarly for lump solutions the kernels are also separable [11] and again we have a closed form of solutions. This is often used as a definition of lump solutions. Further in an analogous way to the construction of n-soliton solutions, one can construct n-lump solutions. Indeed this is possible from a Bäcklund transformation as shown by Kaup [13].

However, soliton solutions in one-dimensional inverse-scattering theories always correspond to a pure bound-state spectrum and there is a relationship between the amplitude and width so that only one is truely independent. For lump solutions this connection is lost and the profile may be quite arbitrary. Lump solutions may be thought of as having more freedom in their profile.

In this paper we investigate features of this three-dimensional three-wave interaction problem from the point of view of the bilinear method and Grammians. The equations and solutions described originally by Kaup can be easily recast in terms of a bilinear formulation. The corresponding equations correspond to lowest-weight equations in the KP three-component hierarchy.

In section 2 we look briefly at the system of equations and the associated linear problem. Singularity analysis is carried out and the resulting expansion allows us to generate the Bäcklund transformation which leads to the bilinear form. The connection between this and Kaup's Bäcklund transformation is noted. In section 3 we consider the 1 and 2 lump solutions obtained by Kaup via the Bäcklund transformation [13] and rewrite these in a Grammian form, which we can then generalize to obtain an n-lump solution. Section 4 gives a direct proof of solutions in the Grammian form. This leads us to examine, in section 5 a set of more general solutions. Finally in section 6 we look at some explicit examples.

2. The three-dimensional three-wave interaction

2.1. The system of equations

The 3D3WR equations [12] take the form

$$\frac{\partial q_i}{\partial X_i} = \gamma_i q_j^* q_k^* \qquad \frac{\partial q_i^*}{\partial X_i} = \gamma_i q_j q_k \tag{2.1}$$

where i, j, k are cyclic and equal to $1, 2, 3$ and $*$ means complex conjugation. The X_i are characteristic coordinates, usually defined by

$$\frac{\partial}{\partial X_i} = -\partial_t - \underline{v}_i \cdot \nabla. \tag{2.2}$$

The γ's are coupling constants and are scaled to unity in magnitude, i.e. $\gamma = \pm 1$, different choices for the γ's will correspond to reflections. We can therefore, without loss of generality, set $\gamma_1 = \gamma_2 = \gamma_3 = 1$. By changing the signs of the fields q_i one of the γ's can always be set equal to +1, it can then be seen that there are really only two distinct cases $\gamma_1 = \gamma_2 = \gamma_3 = 1$ and $\gamma_1 = \gamma_2 = -\gamma_3 = 1$.

Although time, t, occurs in the system we shall generally be working with the characteristic coordinates, thus we may think of this system as a stationary system in three-dimensional space represented by these characteristic coordinates. Alternatively, we can consider the solution of the system at a point in time to be a cross section through the three-dimensional space picture. As time progresses this cross section moves, $t \to -\infty$ corresponding to the characteristic coordinates $X_i \to +\infty$ and $t \to \infty$ corresponding to $X_i \to -\infty$.

The basic scattering problem [12] is given by

$$\frac{\partial \zeta_i}{\partial X_k} = q_j^* \zeta_k \qquad \frac{\partial \zeta_k}{\partial X_i} = q_j \zeta_i \tag{2.3}$$

where i, j, k take cyclic values over $1, 2, 3$. The integrability condition for (2.3) is the original nonlinear system (2.1). Kaup points out [12] that this scattering problem is unusual in that there is no eigenvalue present, thus there are no bound states, so solitons as understood for one-dimensional systems do not occur. However, localized solutions do occur, and are referred to as lumps rather than solitons.

2.2. Singularity analysis

For this system the Painlevé analysis can be carried out. We recall that a partial differential equation (PDE) possesses the Painlevé property when its solutions are single valued about the movable singularity manifold [18].

In order to perform the Painlevé analysis we start from the initial system (2.1) and make the ansatz that the variables q_i, q_i^* can be expanded about the singularity manifold $\varphi(X_1, X_2, X_3) = 0$ as

$$q_i = \sum_{m=0}^{\infty} q_{im} \varphi^{m+\alpha_i} \tag{2.4}$$

$$q_i^* = \sum_{m=0}^{\infty} q^*{}_{im} \varphi^{m+\beta_i} \tag{2.5}$$

where φ, q_{im} and $q^*{}_{im}$ are all analytic functions of X_1, X_2, X_3 in the neighbourhood of $\varphi = 0$ and α_i, β_i are integers. A leading order analysis provides

$$\alpha_i = \beta_i = -1 \tag{2.6}$$

for $i = 1, 2, 3$. The resonances, that is powers of m at which arbitary functions enter into the series, can be calculated. They are

$$m = -1, 0, 2, 3 \tag{2.7}$$

with 0 and 2 repeated twice. The resonance at $m = -1$ represents the arbitrariness of the singularity manifold $\varphi(X_1, X_2, X_3) = 0$. While the 'double' resonance at $m = 0$ is associated with the introduction of two arbitary functions at the lowest level. With some further checks on consistency it can be shown that the Painlevé property is satisfied and consequently the equation is expected to be integrable (see appendix for more details).

To generate the Bäcklund transformation and the bilinear form of the equations, we truncate the series (2.4) and (2.5) at the constant level term [6, 7], that is set

$$q_{im} = q^*{}_{im} = 0 \qquad \text{for } m \geqslant 2. \tag{2.8}$$

Thus we have

$$q_i = \frac{q_{i0}}{\varphi} + q_{i1} \tag{2.9}$$

$$q_i^* = \frac{q^*{}_{i0}}{\varphi} + q^*{}_{i1} \tag{2.10}$$

where the q_{i1} and $q^*{}_{i1}$ satisfy the original equations (2.1),

$$\frac{\partial q_{i1}}{\partial X_i} = q_{j1}^* q_{k1}^* \qquad \frac{\partial q_{i1}^*}{\partial X_i} = q_{j1} q_{k1} \tag{2.11}$$

and

$$q_{i0}\frac{\partial\varphi}{\partial X_i} = -q^*_{j0}q^*_{k0} \qquad q^*_{i0}\frac{\partial\varphi}{\partial X_i} = -q_{j0}q_{k0} \tag{2.12}$$

$$\frac{\partial q_{i0}}{\partial X_i} = (q^*_{j0}q^*_{k1} + q^*_{k0}q^*_{j1}) \qquad \frac{\partial q^*_{i0}}{\partial X_i} = (q_{j0}q_{k1} + q_{k0}q_{j1}) \tag{2.13}$$

for $i, j, k = 1, 2, 3$ and cyclic. Starting with a solution q_{i1}, q^*_{i1} we can generate another solution q_i, q^*_i via equations (2.9) and (2.10), as long as equations (2.11)–(2.13) hold. Thus the above equations (2.9)–(2.13) represent a Bäcklund transformation for the system.

The Bäcklund transformation for this system has previously been written down by Kaup [13]. Given a solution q_i, to generate another solution $\bar{q}_i$, we use

$$\bar{q}_i = q_i + \frac{\zeta^*_k\zeta_j}{D} \tag{2.14}$$

where the function D exists and

$$\partial_i D = \frac{\partial D}{\partial X_i} = -\zeta^*_i\zeta_i. \tag{2.15}$$

The ζ's come from the scattering problem for the original equation

$$\partial_k\zeta_i = q^*_j\zeta_k \tag{2.16}$$

$$\partial_i\zeta_k = q_j\zeta_i \tag{2.17}$$

where, as before, the i, j, k are cyclic. This Bäcklund transformation is indeed the same as the one generated by truncating the series expansion if we make the identifications

$$\begin{aligned} &\varphi \to D \\ &q_{i0} \to \zeta_j\zeta^*_k \\ &q^*_{i0} \to \zeta_k\zeta^*_j. \end{aligned} \tag{2.18}$$

2.3. Bilinearization

We can use the Painlevé analysis as a guide to bilinearization by considering the vacuum solution [6]

$$q_{i1} = q^*_{i1} = 0 \tag{2.19}$$

in the Bäcklund transformation (2.9) and (2.10) to give

$$q_i = \frac{q_{i0}}{\varphi} \tag{2.20}$$

$$q^*_i = \frac{q^*_{i0}}{\varphi}. \tag{2.21}$$

Assuming φ to be real and substituting equations (2.20) and (2.21) into (2.1) and making use of Hirota bilinear operators [9], we obtain the following Hirota bilinear form

$$D_{X_i}q_{i0}\cdot\varphi = q^*_{j0}q^*_{k0} \tag{2.22}$$

$$D_{X_i}q^*_{i0}\cdot\varphi = q_{j0}q_{k0}. \tag{2.23}$$

The D_{X_i} are standard Hirota bilinear operators

$$D_X a\cdot b = \frac{\partial a}{\partial X}b - a\frac{\partial b}{\partial X} = a_X b - ab_X. \tag{2.24}$$

Instead of using the Painlevé analysis as a guidance to the bilinearization we can directly bilinearize this system by choosing the following change of variables:

$$q_i = \frac{G_i}{F} \qquad q_i^* = \frac{G_i^*}{F} \tag{2.25}$$

where F is understood to be real. This recasts our equations as

$$D_{X_i} F \cdot G_i = -G_j^* G_k^* \qquad D_{X_i} F \cdot G_i^* = -G_j G_k \tag{2.26}$$

where i, j, k are cyclic permutations of $1, 2, 3$.

It should perhaps be pointed out at this stage that these equations come straight from the lowest equations in the three-component KP hierarchy, hence integrability is to be expected. The form of solution will be the τ-functions from this hierarchy. These in general will take the form of three-component Wronskians or three-component Grammians. We will take an explicit look at this later.

3. Solutions

Let us consider the simplest type of solution to such a system. Consider the specific case where there is only one wave envelope present, q_1 say, with $q_2 = q_3 = 0$, here the equations reduce to

$$\frac{\partial q_1}{\partial X_1} = 0 \qquad \frac{\partial q_1^*}{\partial X_1} = 0 \tag{3.1}$$

this says that q_1 is independent of X_1 and has arbitrary dependence on X_2 and X_3, thus, to use the language of Kaup we can think of q_1 as being a 'tube' or cylinder extending in the X_1 direction with some profile in the X_2 and X_3 directions.

In general, however, all three fields will be present in the solution, and they can be thought of as 'tubes' lying in the three characteristic directions. In regions where there is an overlap of these 'tubes' interactions will occur, however, as the $X_i \to \pm\infty$ the envelopes will separate and the solutions will cease from changing. Kaup looked at such solutions via inverse-scattering methods [12, 11, 13].

3.1. 1-lump solution

3.1.1. Non-degenerate kernel case. To generate the 1-lump solution we start with the trivial solution $q_i = 0, \forall i = 1, 2, 3$. We then use the Bäcklund transformation [13] to generate another solution, solving first for the ζ's, then calculating D. We have

$$\zeta_i = g_i(X_i) \tag{3.2}$$

$$D = \beta + \sum_{i=1}^{3} \Phi_i(X_i). \tag{3.3}$$

where β is a real constant and

$$\Phi_i(X_i) = \int_{X_i}^{\infty} g_i^*(u) g_i(u)\, \mathrm{d}u \tag{3.4}$$

the g_i's are arbitrary functions of the single variables X_i. Without loss of generality we may set $\beta = 1$, the 1-lump solution is then given by

$$\bar{q_j} = \frac{g_i^* g_k}{1 + \sum_{m=1}^{3} \Phi_m(X_m)}. \tag{3.5}$$

One of the aims of this paper is to relate these solutions to Grammians, in a similar way to Gilson and Nimmo's proceedure for the Davey–Stewartson equation [8]. The Grammian approach being compact and direct, allows a wide class of solutions to be determined. Here we can write the solution in the following form

$$q_i = \frac{G_i}{F} \qquad q_i^* = \frac{G_i^*}{F} \tag{3.6}$$

with

$$F = 1 + \sum_{m=1}^{3} \Phi_m(X_m) = |I + H\Phi| = |\mathcal{F}| \tag{3.7}$$

where I is the (3×3) identity matrix,

$$\Phi = \mathrm{diag}(\Phi_1, \Phi_2, \Phi_3) \qquad H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{3.8}$$

The G's can be expressed as bordered determinants

$$G_1 = -\begin{vmatrix} 0 & 0 & 0 & g_3^* \\ g_2 & & & \\ g_2 & & \mathcal{F} & \\ g_2 & & & \end{vmatrix} = -\begin{vmatrix} 0 & \underline{g_3^\dagger} \\ H\underline{g_2} & \mathcal{F} \end{vmatrix} \tag{3.9}$$

$$G_2 = -\begin{vmatrix} 0 & g_1^* & 0 & 0 \\ g_3 & & & \\ g_3 & & \mathcal{F} & \\ g_3 & & & \end{vmatrix} = -\begin{vmatrix} 0 & \underline{g_1^\dagger} \\ H\underline{g_3} & \mathcal{F} \end{vmatrix} \tag{3.10}$$

$$G_3 = -\begin{vmatrix} 0 & 0 & g_2^* & 0 \\ g_1 & & & \\ g_1 & & \mathcal{F} & \\ g_1 & & & \end{vmatrix} = -\begin{vmatrix} 0 & \underline{g_2^\dagger} \\ H\underline{g_1} & \mathcal{F} \end{vmatrix}. \tag{3.11}$$

Similarly we have

$$G_i^* = -\begin{vmatrix} 0 & \underline{g_j^\dagger} \\ H\underline{g_k} & \mathcal{F} \end{vmatrix} \tag{3.12}$$

for i, j, k cyclic. Here

$$\underline{g_1} = (g_1, 0, 0)^T \qquad \underline{g_2} = (0, g_2, 0)^T \qquad \underline{g_3} = (0, 0, g_3)^T \tag{3.13}$$

$$\underline{g_1^\dagger} = (g_1^*, 0, 0) \qquad \underline{g_2^\dagger} = (0, g_2^*, 0) \qquad \underline{g_3^\dagger} = (0, 0, g_3^*). \tag{3.14}$$

3.1.2. Degenerate kernel case. In addition to this solution (the non-degenerate kernel case) there is, what is called the degenerate kernel case [12]. For the degenerate kernel case the solution is given by,

$$q_j = g_i^*(X_i) g_k(X_k) \frac{1 - \Phi_j(X_j)}{D(X_i, X_j, X_k)} \tag{3.15}$$

where

$$\Phi_i(X_i) = \int_{X_i}^{\infty} g_i^*(u) g_i(u)\, \mathrm{d}u \tag{3.16}$$

and

$$D = 1 - \Phi_i\Phi_j - \Phi_j\Phi_k - \Phi_k\Phi_i + 2\Phi_i\Phi_j\Phi_k. \tag{3.17}$$

D appears in the denominator of the solution, thus, for non-singular solutions, it can be seen from the signs of the coefficients occurring in D that we cannot take the g's as exponentials. Cornille [3] has shown that these degenerate kernels can admit localized solutions as long as the q_i, $i = 1, 2, 3$ are localized and the Φ's are always strictly less than 1.

Again this can be cast in terms of Grammians as follows

$$q_i = \frac{G_i}{F} \tag{3.18}$$

with

$$F = |I + H\Phi| = |\mathcal{F}| \tag{3.19}$$

where I is the (3×3) identity matrix and

$$\Phi = \operatorname{diag}(\Phi_1, \Phi_2, \Phi_3) \qquad H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{3.20}$$

Thus the structure of the solution (3.19) has not changed, only the form of the matrix H occuring. The G_i can be expressed as bordered determinants, just as before, but now with the modified H. For example

$$G_1 = -\begin{vmatrix} 0 & 0 & 0 & g_3^* \\ g_2 & & & \\ 0 & & \mathcal{F} & \\ g_2 & & & \end{vmatrix} = -\begin{vmatrix} 0 & \dfrac{g_3^\dagger}{\mathcal{F}} \\ H\underline{g_2} & \end{vmatrix} \tag{3.21}$$

i.e.

$$G_i = -\begin{vmatrix} 0 & \dfrac{g_k^\dagger}{\mathcal{F}} \\ H\underline{g_j} & \end{vmatrix} \qquad G_i^* = -\begin{vmatrix} 0 & \dfrac{g_j^\dagger}{\mathcal{F}} \\ H\underline{g_k} & \end{vmatrix}. \tag{3.22}$$

3.2. 2-lump solution

Again using the Bäcklund transformation, a 2-lump solution can be generated from the 1-lump solution. For convenience, especially later on in the paper we shall adapt notation. Let us rewrite the 1-lump solution:

$$\zeta_1 = \phi(X_1) \qquad \zeta_2 = \psi(X_2) \qquad \zeta_3 = \sigma(X_3) \tag{3.23}$$

$$\Phi_{11} = \int_{X_1}^{\infty} \phi^*\phi(u)\,\mathrm{d}u \qquad \Psi_{11} = \int_{X_2}^{\infty} \psi^*\psi(u)\,\mathrm{d}u \qquad \Sigma_{11} = \int_{X_3}^{\infty} \sigma^*\sigma(u)\,\mathrm{d}u \tag{3.24}$$

with

$$D_{11} = 1 + \Phi_{11}(X_1) + \Psi_{11}(X_2) + \Sigma_{11}(X_3). \tag{3.25}$$

The 1-lump solution is then given by

$$q_1 = \frac{\psi\sigma^*}{D_{11}} \qquad q_2 = \frac{\sigma\phi^*}{D_{11}} \qquad q_3 = \frac{\phi\psi^*}{D_{11}} \tag{3.26}$$

$$q_1^* = \frac{\psi^*\sigma}{D_{11}} \qquad q_2^* = \frac{\sigma^*\phi}{D_{11}} \qquad q_3^* = \frac{\phi^*\psi}{D_{11}}. \tag{3.27}$$

Also we will introduce

$$D_{ij} = \beta_{ij} + \Phi_{ij}(X_1) + \Psi_{ij}(X_2) + \Sigma_{ij}(X_3) \tag{3.28}$$

where β_{ij} are constants and

$$\Phi_{ij} = \int_{X_1}^{\infty} \phi_i^*\phi_j(u)\,\mathrm{d}u \qquad \Psi_{ij} = \int_{X_2}^{\infty} \psi_i^*\psi_j(u)\,\mathrm{d}u \qquad \Sigma_{ij} = \int_{X_3}^{\infty} \sigma_i^*\sigma_j(u)\,\mathrm{d}u. \tag{3.29}$$

For the 2-lump solution we let $i, j = 1, 2$ and can without loss of generality set $\beta_{11} = \beta_{22} = 1$. Then

$$q_i = \frac{G_i}{F} \tag{3.30}$$

with

$$\begin{aligned} F &= D_{11}D_{22} - D_{12}D_{21} \\ G_1 &= D_{22}\psi_1\sigma_1^* + D_{12}\psi_1\sigma_2^* + D_{21}\psi_2\sigma_1^* + D_{11}\psi_2\sigma_2^* \\ G_2 &= D_{22}\sigma_1\phi_1^* + D_{12}\sigma_1\phi_2^* + D_{21}\sigma_2\phi_1^* + D_{11}\sigma_2\phi_2^* \\ G_3 &= D_{22}\phi_1\psi_1^* + D_{12}\phi_1\psi_2^* + D_{21}\phi_2\psi_1^* + D_{11}\phi_2\psi_2^*. \end{aligned} \tag{3.31}$$

This solution can also be recast as a Grammian, this will now be a 6×6 determinant.

$$q_i = \frac{G_i}{F} \tag{3.32}$$

with $F = |I + H\Phi| = |\mathcal{F}|$, where I is the (6×6) identity matrix, Φ is a matrix with on-diagional (2×2) blocks and zeros elsewhere, H is a block matrix with nine identical (2×2) blocks.

$$\Phi = \begin{pmatrix} \Phi_{ij} & & 0 \\ & \Psi_{ij} & \\ 0 & & \Sigma_{ij} \end{pmatrix} \qquad H = \begin{pmatrix} B & B & B \\ B & B & B \\ B & B & B \end{pmatrix} \tag{3.33}$$

with

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}. \tag{3.34}$$

The G_i's can be expressed as a bordered determinants

$$G_1 = -\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\psi} & \mathcal{F} \end{vmatrix} \tag{3.35}$$

$$G_2 = -\begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\sigma} & \mathcal{F} \end{vmatrix} \tag{3.36}$$

$$G_3 = -\begin{vmatrix} 0 & \underline{\psi}^\dagger \\ H\underline{\phi} & \mathcal{F} \end{vmatrix} \tag{3.37}$$

where

$$\underline{\phi} = (\phi_1, \phi_2, 0, 0, 0, 0)^T \qquad \underline{\psi} = (0, 0, \psi_1, \psi_2, 0, 0)^T \qquad \underline{\sigma} = (0, 0, 0, 0, \sigma_1, \sigma_2)^T \tag{3.38}$$

$$\underline{\phi}^\dagger = (\phi_1^*, \phi_2^*, 0, 0, 0, 0) \qquad \underline{\psi}^\dagger = (0, 0, \psi_1^*, \psi_2^*, 0, 0) \qquad \underline{\sigma}^\dagger = (0, 0, 0, 0, \sigma_1^*, \sigma_2^*). \tag{3.39}$$

3.3. n-lump solution

The reason that we have taken time to change the notation is that now it is possible to postulate a simple form for the n-lump solution which has the structure of a three-component Grammian.

$$q_i = \frac{G_i}{F} \tag{3.40}$$

with

$$F = |I + H\Phi| = |\mathcal{F}| \tag{3.41}$$

where I is the $(3n \times 3n)$ identity matrix, Φ is a matrix with on-diagional $(n \times n)$ blocks and zeros elsewhere, H is a block matrix with nine identical $(n \times n)$ blocks.

$$\Phi = \begin{pmatrix} \Phi_{ij} & & 0 \\ & \Psi_{ij} & \\ 0 & & \Sigma_{ij} \end{pmatrix} \qquad H = \begin{pmatrix} B & B & B \\ B & B & B \\ B & B & B \end{pmatrix} \tag{3.42}$$

with

$$B_{ij} = \beta_{ij} \qquad i, j = 1 \ldots n. \tag{3.43}$$

The G's can be expressed as bordered determinants

$$G_1 = -\begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\psi} & \overline{\mathcal{F}} \end{vmatrix} \tag{3.44}$$

$$G_2 = -\begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\sigma} & \overline{\mathcal{F}} \end{vmatrix} \tag{3.45}$$

$$G_3 = -\begin{vmatrix} 0 & \underline{\psi}^\dagger \\ H\underline{\phi} & \overline{\mathcal{F}} \end{vmatrix} \tag{3.46}$$

where

$$\underline{\phi} = (\phi_1, \ldots \phi_n; 0, \ldots 0; 0, \ldots 0)^T \tag{3.47}$$

$$\underline{\psi} = (0, \ldots 0; \psi_1, \ldots \psi_n; 0, \ldots 0)^T \tag{3.48}$$

$$\underline{\sigma} = (0, \ldots 0; 0, \ldots 0; \sigma_1, \ldots \sigma_n)^T \tag{3.49}$$

$$\underline{\phi}^\dagger = (\phi_1^*, \ldots \phi_n^*; 0, \ldots 0; 0, \ldots 0) \tag{3.50}$$

$$\underline{\psi}^\dagger = (0, \ldots 0; \psi_1^*, \ldots \psi_n^*; 0, \ldots 0) \tag{3.51}$$

$$\underline{\sigma}^\dagger = (0, \ldots 0; 0, \ldots 0; \sigma_1^*, \ldots \sigma_n^*). \tag{3.52}$$

4. Direct proof of the solution

For a direct proof of the solution we use a Jacobi identity [2]. The basic Jacobi identity is given as follows: consider an $N \times N$ matrix A we write $A^{i,\ldots,j}_{k,\ldots,l}$ for the minor obtained by omitting the ith, $\ldots$, jth rows and the kth, $\ldots$, lth columns, in which notation the Jacobi identity is

$$|A|A^{i,j}_{k,l} = \begin{vmatrix} A^i_k & A^j_k \\ A^i_l & A^j_l \end{vmatrix}. \tag{4.1}$$

Here we need the following form of the identity:

$$|\mathcal{F}| \begin{vmatrix} 0 & 0 & \underline{\phi}^\dagger \\ 0 & 0 & \underline{\sigma}^\dagger \\ H\underline{\phi} & H\underline{\psi} & \overline{\mathcal{F}} \end{vmatrix} = \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\phi} & \overline{\mathcal{F}} \end{vmatrix} \begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\psi} & \overline{\mathcal{F}} \end{vmatrix} - \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\psi} & \overline{\mathcal{F}} \end{vmatrix} \begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\phi} & \overline{\mathcal{F}} \end{vmatrix}. \tag{4.2}$$

We can show that the derivatives we require can all be expressed in terms of bordered determinants. These expressions arise because, in general, for an $N \times N$ matrix A whose entries a_{ij} are such that

$$\frac{\partial a_{ij}}{\partial X} = \alpha_i \beta_j \tag{4.3}$$

the derivative of the determinant can be written as

$$\frac{\partial |A|}{\partial X} = \sum_{i,j=1}^{N} (-1)^{i+j} \alpha_i \beta_j A^i_j = - \begin{vmatrix} 0 & \beta_1 & \dots & \beta_N \\ \alpha_1 & & & \\ \vdots & & A & \\ \alpha_N & & & \end{vmatrix}. \tag{4.4}$$

Thus, we can show that

$$\frac{\partial F}{\partial X_1} = \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\phi} & \overline{\mathcal{F}} \end{vmatrix} \tag{4.5}$$

and

$$\frac{\partial G_1}{\partial X_1} = - \begin{vmatrix} 0 & 0 & \underline{\phi}^\dagger \\ 0 & 0 & \underline{\sigma}^\dagger \\ H\underline{\phi} & H\underline{\psi} & \overline{\mathcal{F}} \end{vmatrix}. \tag{4.6}$$

Thus (4.2) is actually

$$-F \frac{\partial G_1}{\partial X_1} = - \frac{\partial F}{\partial X_1} G_1 - G^*_2 G^*_3 \tag{4.7}$$

this is precisely the bilinear equation

$$D_{X_1} F \cdot G_1 = -G^*_2 G^*_3. \tag{4.8}$$

Similar identities to (4.2) exist, these give us the other bilinear equations.

5. A more general solution

The proof that the n-lump solutions satisfy the equations, actually applies more generally. The proof will still hold for any constant Hermitian matrix you may wish to take for H. In general, the form $F = |I + H\Phi|$, where Φ could be a block matrix where each block may not necessarily be of the same size will still be a solution. Thus

$$q_i = \frac{G_i}{F} \qquad q^*_i = \frac{G^*_i}{F} \tag{5.1}$$

will be a solution of the system, with

$$F = |I + H\Phi| = |\mathcal{F}| \tag{5.2}$$

where I is the $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ identity matrix, H is a Hermitian matrix, Φ is an $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ matrix with non-zero on-diagional blocks and zeros elsewhere

$$\Phi = \begin{pmatrix} \Phi_{ij} & & 0 \\ & \Psi_{kl} & \\ 0 & & \Sigma_{mn} \end{pmatrix} \qquad \begin{matrix} i, j = 1 \dots n_1 \\ k, l = 1 \dots n_2 \\ m, n = 1 \dots n_3. \end{matrix} \tag{5.3}$$

The G's can be expressed as bordered determinants

$$G_1 = - \begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\psi} & \overline{\mathcal{F}} \end{vmatrix} \qquad G^*_1 = - \begin{vmatrix} 0 & \underline{\psi}^\dagger \\ H\underline{\sigma} & \overline{\mathcal{F}} \end{vmatrix} \tag{5.4}$$

$$G_2 = - \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\sigma} & \overline{\mathcal{F}} \end{vmatrix} \qquad G^*_2 = - \begin{vmatrix} 0 & \underline{\sigma}^\dagger \\ H\underline{\phi} & \overline{\mathcal{F}} \end{vmatrix} \tag{5.5}$$

$$G_3 = - \begin{vmatrix} 0 & \underline{\psi}^\dagger \\ H\underline{\phi} & \overline{\mathcal{F}} \end{vmatrix} \qquad G^*_3 = - \begin{vmatrix} 0 & \underline{\phi}^\dagger \\ H\underline{\psi} & \overline{\mathcal{F}} \end{vmatrix} \tag{5.6}$$

where

$$\underline{\phi} = (\phi_1, \ldots \phi_{n_1}; 0, \ldots 0; 0, \ldots 0)^T \tag{5.7}$$

$$\underline{\psi} = (0, \ldots 0; \psi_1, \ldots \psi_{n_2}; 0, \ldots 0)^T \tag{5.8}$$

$$\underline{\sigma} = (0, \ldots 0; 0, \ldots 0; \sigma_1, \ldots \sigma_{n_3})^T \tag{5.9}$$

$$\underline{\phi}^\dagger = (\phi_1^*, \ldots \phi_{n_1}^*; 0, \ldots .0; 0, \ldots 0) \tag{5.10}$$

$$\underline{\psi}^\dagger = (0, \ldots 0; \psi_1^*, \ldots \psi_{n_2}^*; 0, \ldots 0) \tag{5.11}$$

$$\underline{\sigma}^\dagger = (0, \ldots 0; 0, \ldots 0; \sigma_1^*, \ldots \sigma_{n_3}^*). \tag{5.12}$$

This is quite a broad class of solutions, however, we have to keep in mind that the F we actually want to take, should be non-zero everywhere, otherwise the solutions will contain singularities.

6. Examples

The essential difference between soliton solutions and lump solutions is the absence of (discrete) spectral parameters in the inverse-scattering theory. This difference manifests itself in the functions ϕ_i, ψ_i and σ_i. In the case of related theories with solitons these functions contain the spectral parameters and are always restricted to obey some linear equation relating derivatives of the different independent variables together, here there is no such restriction. In this section we shall look at solutions to some simple cases.

6.1. The (1, 1, 1) case

Evaluating F in (5.2) gives

$$\begin{aligned} F = 1 + h_{11}\Phi_{11} + h_{22}\Psi_{11} + h_{33}\Sigma_{11} + \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \Phi_{11}\Psi_{11} + \begin{vmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{vmatrix} \Phi_{11}\Sigma_{11} \\ + \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} \Psi_{11}\Sigma_{11} + \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} \Phi_{11}\Psi_{11}\Sigma_{11} \end{aligned} \tag{6.1}$$

and from (5.4)

$$G_1 = \psi_1\sigma_1^* \left(h_{32} + \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix} \Phi_{11} \right) \tag{6.2}$$

with the Φ_{11}, Ψ_{11}, Σ_{11} given by (3.24), $n_1 = n_2 = n_3 = 1$ and H as

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix}. \tag{6.3}$$

6.1.1. Localized profiles. If ϕ_1, ψ_1 and σ_1 are taken to be localized functions, for instance Lorentzians or Gaussians, their asymptotic behaviour is simple to describe. As

$$\begin{aligned} &X_1 \to +\infty \qquad \Phi_{11} \to 0 \qquad \phi_1 \to 0 \qquad \phi_1^* \to 0 \\ &X_1 \to -\infty \qquad \Phi_{11} \to \Phi_{-\infty} \qquad \phi_1 \to 0 \qquad \phi_1^* \to 0 \end{aligned} \tag{6.4}$$

where $\Phi_{-\infty}$ is a constant. Similar sort of behaviour occurs for Ψ_{11} and Σ_{11} as $X_2, X_3 \to \pm\infty$. As $X_1 \to \infty$

$$F \to 1 + h_{22}\Psi_{11} + h_{33}\Sigma_{11} + \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} \Psi_{11}\Sigma_{11} \tag{6.5}$$

$$G_1 \to h_{32}\psi_1\sigma_1^* \tag{6.6}$$

as $X_1 \to -\infty$

$$F \to a + b\Psi_{11} + c\Sigma_{11} + d\Psi_{11}\Sigma_{11} \tag{6.7}$$

$$G_1 \to e\psi_1\sigma_1^* \tag{6.8}$$

where a, b, c, d, e are constants

$$a = 1 + h_{11}\Phi_{-\infty} \tag{6.9}$$

$$b = h_{22} + \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \Phi_{-\infty} \qquad c = h_{33} + \begin{vmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{vmatrix} \Phi_{-\infty} \tag{6.10}$$

$$d = \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} + \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} \Phi_{-\infty} \tag{6.11}$$

$$e = h_{32} + \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix} \Phi_{-\infty}. \tag{6.12}$$

We can choose ϕ, ψ, σ for instance as,

$$\phi_1 = c_1 \mathrm{e}^{-(X_1 - X_1^0)^2} \qquad \psi_1 = c_2 \mathrm{e}^{-(X_2 - X_2^0)^2} \qquad \sigma_1 = c_3 \mathrm{e}^{-(X_3 - X_3^0)^2} \tag{6.13}$$

where $c_1, c_2, c_3, X_1^0, X_2^0, X_3^0$ are real constants. The q_1-field will take the form of a filled in 'tube' in three-dimensional space in the direction of the X_1-axis, asymptotically for large X_1 the cross section profile will be determined by the functions ψ_1, σ_1. In this particular case we obtain a lump. Similarly at X_1 large and negative the profile will again be a lump of different size. The 'interaction' region can be considered to be the region near the intersection of the coordinate axes. In figure 1, we show q_1 plotted in the X_2X_3-plane for fixed X_1 with

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}. \tag{6.14}$$

As, in general the cross section profile of a $(1, 1, 1)$-solution is arbitrary, by choosing the ϕ_1, ψ_1 and σ_1's differently this simple solution could have a much more interesting appearence, however, the basic character of the 'tube' structure in the X_1 direction will not change. This behaviour is similar to the lumps as described by Kaup [13].

6.1.2. Non-localized profiles. As an alternative to choosing our functions ϕ_1, ψ_1, σ_1 as above we could choose functions which don't decay as the $X_i \to \pm\infty$, we shall look at exponentials, we would expect the behaviour here to mimic more closely that of solitons. Choose

$$\phi_1 = \mathrm{e}^{-p(X_1 - p)} \qquad \psi_1 = \mathrm{e}^{-q(X_2 - q)} \qquad \sigma_1 = \mathrm{e}^{-r(X_3 - r)}. \tag{6.15}$$

With the way our boundary conditions have been set up earlier it is necessary to choose $\mathrm{Re}\,(p), \mathrm{Re}\,(q), \mathrm{Re}\,(r) > 0$ (however, with some reconsideration of the boundary conditions these could be taken negative). Now the choice of H in our solution is more critical, we

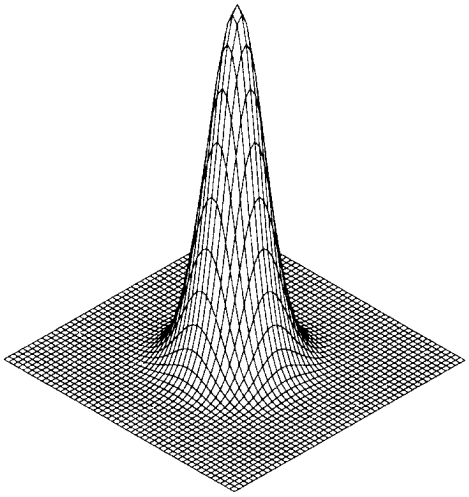

Figure 1. The q_1-field, plotted in the X_2X_3-plane for fixed $X_1 = -5$, $\phi_1 = \exp(-X_1^2)/2$, $\psi_1 = \exp(-X_2^2)/2$, $\sigma_1 = \exp(-X_3^2)/2$.

can obtain solutions that are localized or are 'ridge' like. For this exponential case the asymptotics are

$$\begin{array}{llll} X_1 \to +\infty & \Phi_{11} \to 0 & \phi_1 \to 0 & \phi_1^* \to 0 \\ X_1 \to -\infty & \Phi_{11} \to \infty & \phi_1 \to \infty & \phi_1^* \to \infty \end{array} \tag{6.16}$$

thus as $X_1 \to +\infty$

$$F \to 1 + h_{22}\Psi_{11} + h_{33}\Sigma_{11} + \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} \Psi_{11}\Sigma_{11} \tag{6.17}$$

$$G_1 \to h_{32}\psi_1\sigma_1^* \tag{6.18}$$

as $X_1 \to -\infty$

$$F \to a + b\Psi_{11} + c\Sigma_{11} + d\Psi_{11}\Sigma_{11} \tag{6.19}$$

$$G_1 \to e\psi_1\sigma_1^* \tag{6.20}$$

where a, b, c, d, e are constants

$$a = h_{11} \qquad b = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \qquad c = \begin{vmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{vmatrix} \tag{6.21}$$

$$d = \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} \qquad e = \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix}. \tag{6.22}$$

If the coefficients of the $\Psi_{11}, \Sigma_{11}, \Psi_{11}\Sigma_{11}$ and the constant term are all present in F the solution will look like a lump in the X_2X_3-plane, for fixed X_1. This lump will change only in the interaction region (around $X_1 = 0$). This would be obtained with an H such as

$$H = \begin{pmatrix} 1 & \alpha & \beta \\ \alpha^* & 1 & \eta \\ \beta^* & \eta^* & 1 \end{pmatrix} \tag{6.23}$$

where $|\alpha|, |\beta|, |\eta|$ are all strictly less than 1 (see figure 2).

In figure 3 a surface of constant density of q_1-field has been plotted in three-dimensional space, this looks like a 'tube' parallel to the X_1-axis.

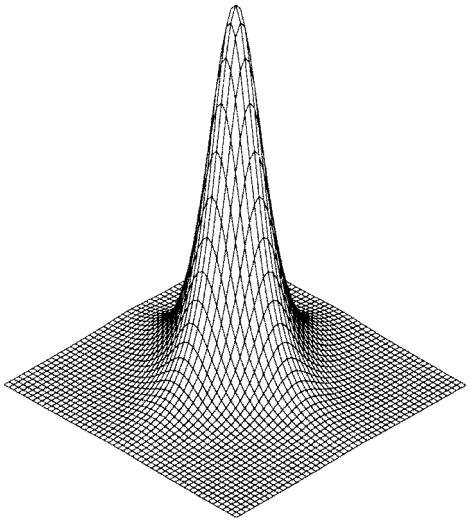

Figure 2. The q_1-field, plotted in the X_2X_3-plane for fixed $X_1 = 10$, $\phi_1 = \exp(-X_1)$, $\psi_1 = \exp(-X_2)$, $\sigma_1 = \exp(-X_3)$, $h_{11} = h_{22} = h_{33} = 1$, $h_{12} = h_{23} = h_{31} = \frac{1}{2}$.

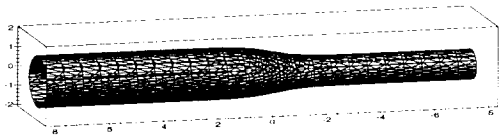

Figure 3. The q_1-field, here we have plotted the surface $|q_1| = 0.13$ in $X_1X_2X_3$-space, $\phi_1 = \exp(-X_1)$, $\psi_1 = \exp(-X_2)$, $\sigma_1 = \exp(-X_3)$, $h_{11} = h_{22} = h_{33} = 1$, $h_{12} = h_{23} = h_{31} = \frac{1}{2}$.

In the case of the original lumps described by Kaup ($h_{ij} = 1, \forall ij = 1, 2, 3$) some of the coefficients of Ψ_{11} etc will be missing, this will have the effect of producing a ridge (see figure 4) as $X_1 \to +\infty$. At $X_1 = -\infty$ this ridge will disappear since

$$e = \begin{vmatrix} h_{11} & h_{12} \\ h_{31} & h_{32} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0. \tag{6.24}$$

It is also possible to set the boundary conditions up differently so as to obtain plane-wave solutions, i.e. solutions where the ridge does not decay to zero in either direction. However, for simplicity we shall not discuss these solutions here.

6.2. The (2,1,1) case

The $(2, 1, 1)$ case is formed by introducing an extra arbitrary function $\phi_2(X_1)$ into the solution. This does not appear to correspond to the solutions discussed by Kaup via inverse-scattering or Bäcklund transformations. The introduction of the function introduces extra features into the solutions. The structure will be reminicant of say the (2,1)-dromion solution in the Davey–Stewartson equation. The function F will now be a 4×4 determinant. As before, the precise terms present will determine the shape of the solution.

6.2.1. Localized profiles. Choosing localized functions

$$\phi_1 = \mathrm{e}^{-(X_1-\alpha)^2} \qquad \phi_2 = \mathrm{e}^{-(X_1-\beta)^2} \qquad \psi_1 = \mathrm{e}^{-(X_2-X_2^0)^2} \qquad \sigma_1 = \mathrm{e}^{-(X_3-X_3^0)^2} \tag{6.25}$$

where α, β, X_2^0, X_3^0 are constants. Looking at the q_2-field we observe, in three-dimensional space, two 'tubes' centred on $(X_1, X_3) = (\alpha, X_3^0)$ and (β, X_3^0) lying in the direction of the

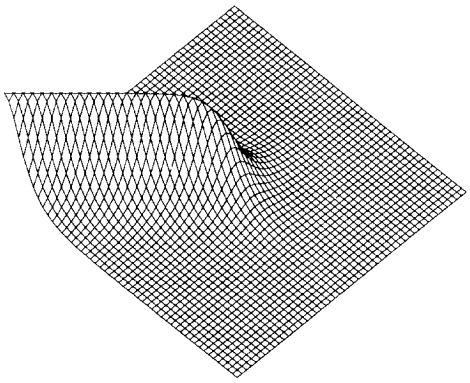

Figure 4. The q_1-field, plotted in the X_2X_3-plane for fixed $X_1 = 10$, $\phi_1 = \exp(-X_1)$, $\psi_1 = \exp(-X_2)$, $\sigma_1 = \exp(-X_3)$, $h_{ij} = 1$ for $ij = 1, 2, 3$.

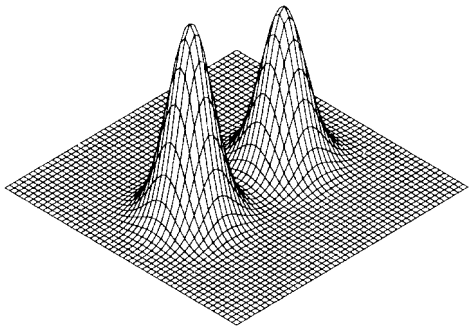

Figure 5. The q_2-field, plotted in the X_1X_3-plane for fixed $X_2 = 10$, $\phi_1 = \exp(-(X_1 - 2)^2)$, $\phi_2 = \exp(-(X_1 + 2)^2)$, $\psi_1 = \exp(-X_2^2)$, $\sigma_1 = \exp(-X_3^2)$, $h_{ij} = 1$ for $i, j = 1, 2, 3$.

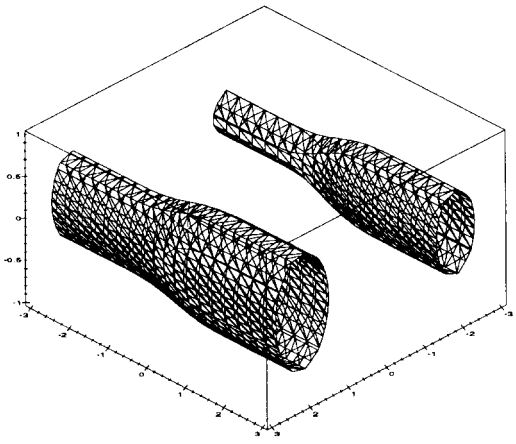

Figure 6. q_2-field, surface $q_2 = 0.12$ plotted in three-dimensional space, $\phi_1 = \exp(-(X_1 - 2)^2)$, $\phi_2 = \exp(-(X_1 + 2)^2)$, $\psi_1 = \exp(-X_2^2)$, $\sigma_1 = \exp(-X_3^2)$, $h_{ij} = 1$ for $i, j = 1, 2, 3$.

X_2-axis. The q_3-field is similar with the 'tubes' lying in the direction of the X_3-axis. The q_1-field has just one 'tube', however, there are interactions around $X_1 = \alpha$ and $X_1 = \beta$. These solutions are shown in figures 5 and 6.

6.2.2. Non-localized profiles. Here as before the values of h_{ij} will determine the kind of solution. Taking exponentials

$$\phi_1 = \mathrm{e}^{-p_1(X_1-p_1)} \qquad \phi_2 = \mathrm{e}^{-p_2(X_1-p_2)} \qquad \psi = \mathrm{e}^{-q(X_2-q)} \qquad \sigma = \mathrm{e}^{-r(X_3-r)} \tag{6.26}$$

and plotting, for instance q_2 we can obtain a solution that appears to be two lumps in the X_1X_3-plane, ie two 'tubes' parallel to the X_2-axes. Plotting q_1 the solution here is one 'tube' parallel to the X_1 axis. This is similar to the localized profiles case. Again by chosing H such that several minors vanish we can also get ridge-type solutions (see figure 7).

6.3. The (l, m, n) case

In general these solutions will take the form of multiple lumps or ridges. The number of lumps will depend on the values of l, m, n. For instance in the least degenerate cases the

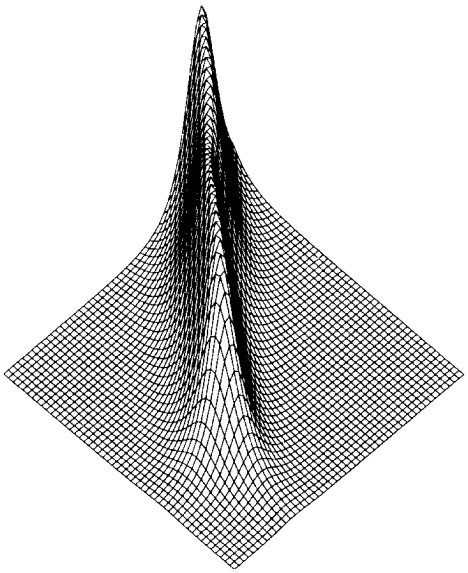

Figure 7. The q_2-field, plotted in the X_1X_3-plane for fixed $X_2 = 10$, $\phi_1 = \exp(-(X_1 - 1))$, $\phi_2 = \exp(-2(X_1 - 2))$, $\psi = \exp(-X_2)$, $\sigma = \exp(-X_3)$, $h_{11} = h_{13} = h_{22} = h_{24} = h_{33} = h_{44} = 1$, $h_{12} = h_{14} = h_{23} = h_{34} = 0$.

q_1-field will have $m \times n$ lumps, if viewed in $X_1 =$ constant plane, with 'interaction' regions occuring at l different values of X_1.

7. Conclusions

The aim of this paper was to investigate a broad class of solutions to the 3D3WR interaction. The types of solutions can be broadly catogorized into two classes, lump solutions and ridge solutions. The lump solutions arise from either choosing our basic functions as localized or by choosing our matrix H so as the coefficients of all the possible terms in F and G are present. The ridge solutions are essentially resonant solutions where by choice of a specific H not all the coefficients are present. The class of solutions presented here includes the 'lump' solutions of Kaup.

Acknowledgments

This work was supported in part by the Engineering and Physical Sciences Research Council. CRG would like to thank Professor J Satsuma for useful comments.

8. Appendix. The Painlevé analysis

For this system the Painlevé analysis can be carried out. We recall that, informally, one says that a PDE possesses the Painlevé property when it's solutions are single valued about the movable singularity manifold [18].

In order to perform the Painlevé analysis we start from the system (2.1)

$$\frac{\partial q_i}{\partial X_i} = q_j^* q_k^* \tag{8.1}$$

$$\frac{\partial q_i^*}{\partial X_i} = q_j q_k \tag{8.2}$$

where i, j, k are cyclic and equal to $1, 2, 3$ and $*$ means complex conjugation. The q_i are functions of the X_j.

Now we make the ansatz that the variables q_i, q_i^* can be expanded about the singularity manifold $\varphi(X_1, X_2, X_3) = 0$ as

$$q_i = \sum_{m=0}^{\infty} u_{im} \varphi^{m+\alpha_i} \tag{8.3}$$

$$q_i^* = \sum_{m=0}^{\infty} v_{im} \varphi^{m+\beta_i} \tag{8.4}$$

where φ, u_{im} and v_{im} are all analytic functions of the X_i, in the neighbourhood of $\varphi = 0$ and α_i, β_i are integers.

Inserting (8.3) and (8.4) into equations (8.1) and (8.2) a leading-order analysis provides

$$\alpha_i = \beta_i = -1 \tag{8.5}$$

for all i, with

$$\frac{\partial \varphi}{\partial X_i} = \frac{-v_{j0} v_{k0}}{u_{i0}} \tag{8.6}$$

$$\frac{\partial \varphi}{\partial X_i} = \frac{-u_{j0} u_{k0}}{v_{i0}}. \tag{8.7}$$

From equations (8.6) and (8.7) we may choose two of the u_{i0}, v_{i0} (u_{30} and v_{10} say) as arbitary functions and then

$$u_{10} = \frac{1}{v_{10}} \frac{\partial \varphi}{\partial X_2} \frac{\partial \varphi}{\partial X_3} \tag{8.8}$$

$$u_{20} = -\frac{v_{10}}{u_{30}} \frac{\partial \varphi}{\partial X_1} \tag{8.9}$$

$$v_{20} = -\frac{u_{30}}{v_{10}} \frac{\partial \varphi}{\partial X_3} \tag{8.10}$$

$$v_{30} = \frac{1}{u_{30}} \frac{\partial \varphi}{\partial X_2} \frac{\partial \varphi}{\partial X_1}. \tag{8.11}$$

We obtain the resonances, that is values of m at which arbitary functions enter into the series, by substituting (8.3) and (8.4) into equations (8.1) and (8.2) retaining leading order terms only. As a result, we obtain the matrix equation

$$[M][V] = 0 \qquad [V]^T = [u_{1m}, u_{2m}, u_{3m}, v_{1m}, v_{2m}, v_{3m}] \tag{8.12}$$

for the lowest-order coefficients, where

$$[M] = \begin{bmatrix} P_1 & 0 & 0 & 0 & -v_{30} & -v_{20} \\ 0 & P_2 & 0 & -v_{30} & 0 & -v_{10} \\ 0 & 0 & P_3 & -v_{20} & -v_{10} & 0 \\ 0 & -u_{30} & -u_{20} & P_1 & 0 & 0 \\ -u_{30} & 0 & -u_{10} & 0 & P_2 & 0 \\ -u_{20} & -u_{10} & 0 & 0 & 0 & P_3 \end{bmatrix} \tag{8.13}$$

with $P_i = (m-1)\frac{\partial \varphi}{\partial X_i}$ for $i = 1, 2, 3$. The resonances are obtained when $\det M = 0$, which yields the resonances $m = -1, 0, 2, 3$ with 0 and 2 repeated twice. The resonance at $m = -1$ represents the arbitariness of the singularity manifold $\varphi(X_1, X_2, X_3) = 0$. While the 'double' resonance at $m = 0$ is associated with the arbitary functions u_{30} and v_{10}.

In order to check the existence of arbitary functions at the other resonances we use the Laurent expansion of (8.3) and (8.4) in equations (8.1) and (8.2). Now collecting the coefficients of $(\varphi^{-1}, \varphi^{-1}, \varphi^{-1}, \varphi^{-1}, \varphi^{-1}, \varphi^{-1})$ we obtain the set of equations

$$\frac{\partial u_{i0}}{\partial X_i} - [v_{j0}v_{k1} + v_{j1}v_{k0}] = 0 \tag{8.14}$$

$$\frac{\partial v_{i0}}{\partial X_i} - [u_{j0}u_{k1} + u_{j1}u_{k0}] = 0. \tag{8.15}$$

Similarily, collecting the coefficients of $(\varphi^0, \varphi^0, \varphi^0, \varphi^0, \varphi^0, \varphi^0)$, we obtain

$$\frac{\partial u_{i1}}{\partial X_i} + u_{i2}\frac{\partial \varphi}{\partial X_i} - [v_{j0}v_{k2} + v_{j2}v_{k0} + v_{j1}v_{k1}] = 0 \tag{8.16}$$

$$\frac{\partial v_{i1}}{\partial X_i} + v_{i2}\frac{\partial \varphi}{\partial X_i} - [u_{j0}u_{k2} + u_{j2}u_{k0} + u_{j1}u_{k1}] = 0. \tag{8.17}$$

Finally at $(\varphi^1, \varphi^1, \varphi^1, \varphi^1, \varphi^1, \varphi^1)$

$$\frac{\partial u_{i2}}{\partial X_i} + 2u_{i3}\frac{\partial \varphi}{\partial X_i} - [v_{j0}v_{k3} + v_{j3}v_{k0} + v_{j1}v_{k2} + v_{j2}v_{k1}] = 0 \tag{8.18}$$

$$\frac{\partial v_{i2}}{\partial X_i} + 2v_{i3}\frac{\partial \varphi}{\partial X_i} - [u_{j0}u_{k3} + u_{j3}u_{k0} + u_{j1}u_{k2} + u_{j2}u_{k1}] = 0. \tag{8.19}$$

To show the above equations have the required number of arbitary functions becomes tedious for the general manifold. We adapt the Kruskal ansatz [17]. By assuming $\varphi(X_1, X_2, X_3) = X_1 + \psi(X_2, X_3)$ the calculations are somewhat simpler.

Solving equations (8.14) and (8.15) we can determine u_{i1} and v_{i1} uniquely, given u_{i0} and v_{i0} $(i = 1, 2, 3)$. Solving (8.16) and (8.17) by repeated substitution we arrive at two arbitary functions, u_{32} and v_{22} say, providing

$$v_{10}\left[\frac{\partial v_{21}}{\partial X_2} + u_{30}\frac{\partial u_{11}}{\partial X_1} - u_{30}v_{21}v_{31} - u_{31}u_{11}\right] - u_{30}v_{20}\left[\frac{\partial u_{21}}{\partial X_2} + v_{30}\frac{\partial v_{11}}{\partial X_1} - u_{21}u_{31}v_{30} - v_{31}v_{11}\right] = 0 \tag{8.20}$$

$$u_{10}\left[\frac{\partial u_{31}}{\partial X_3} + v_{20}\frac{\partial v_{11}}{\partial X_1} - v_{20}u_{21}u_{31} - v_{11}v_{21}\right] - u_{30}v_{20}\left[\frac{\partial v_{31}}{\partial X_3} + u_{20}\frac{\partial u_{11}}{\partial X_1} - u_{20}v_{21}v_{31} - u_{11}u_{21}\right] = 0 \tag{8.21}$$

are both satisfied. With a little algebra this is shown to be the case. So the 'double' resonance at $m = 2$ corresponds to the arbitariness of u_{32} and v_{22}, with u_{12}, u_{22}, v_{12} and v_{32} in terms of previously determined functions.

For (8.18) and (8.19) we find that v_{33} is arbitary providing the following is true

$$g3 + \frac{u_{20}f_1}{2} + \frac{u_{20}v_{30}}{3\partial\varphi/\partial X_2}\left[g_2 + \frac{u_{30}f_1}{2}\right] + \frac{2u_{10}}{3\partial\varphi/\partial X_2}\left[f_2 + \frac{v_{30}g_1}{2}\right] - \frac{v_{20}}{3} \times\left[f_2 + \frac{v_{30}g_1}{2}\right] - \frac{\partial\varphi/\partial X_2}{u_{30}^2}\left[f_3 + \frac{v_{20}g_1}{2} + \frac{2v_{10}}{3\partial\varphi/\partial X_2}\left[g_2 + \frac{u_{30}f_1}{2}\right]\right] = 0 \tag{8.22}$$

with $f_i = \frac{\partial u_{i2}}{\partial X_i} - v_{j1}v_{k2} - v_{j2}v_{k1}$ and $g_i = \frac{\partial v_{i2}}{\partial X_i} - u_{j1}u_{k2} - u_{j2}u_{k1}$. Again this can be shown to be the case.

So the solution q_i, q_i^* of equations (8.3) and (8.4) admits the required number of arbitrary functions without the introduction of movable critical manifolds. Hence the Painlevé property is satisfied for (8.1) and (8.2).

References

[1] Ablowitz M J and Haberman R 1975 Resonantly coupled nonlinear evolution equations *J. Math. Phys.* **16** 2301–5

[2] Aitken A C 1956 *Determinants and Matrices* University Mathematical Texts. Oliver and Boyd (New York: Interscience)

[3] Cornille H 1979 Solutions of the nonlinear 3-wave equations in 3 spatial dimensions *J. Math. Phys.* **20** 1653–66

[4] Craik A D D 1978 Evolution in space and time of resonant wave triads II. A class of exact solutions *Proc. R. Soc.* A **363** 257–69

[5] Craik A D D 1985 *Three Wave Resonance* (Cambridge: Cambridge University Press)

[6] Ganesan S and Lakshmanan M 1987 Singularity analysis and Hirota's bilinearistaion of the DS equation *J. Phys. A: Math. Gen.* **20** L1143–7

[7] Gibbon J D, Radmore P, Tabor M and Wood D 1985 The Painlevé property and Hirota's method *Stud. Appl. Math.* **72** 39–63

[8] Gilson C R and Nimmo J J C 1991 A direct method for dromion solutions of the Davey–Stewartson equation and their asymptotic properties *Proc. R. Soc.* A **435** 339–57

[9] Hirota R 1980 Direct methods in soliton theory *Solitons* (Berlin: Springer) pp 157–76

[10] Kaup D J 1976 The 3-wave interaction—A nondispersive phenomenon *SIAM* **55** 9–44

[11] Kaup D J 1980 The inverse scattering solution for the full three-dimensional three-wave resonant interaction *Physica* **1D** 45–67

[12] Kaup D J 1980 A method for solving the separable initial value problem of the full three-dimensional three-wave interaction *SIAM* **62** 75–83

[13] Kaup D J 1981 The lump solutions and the Bäcklund transformation for the full three-dimensional three-wave resonant interaction *J. Math. Phys.* **22** 1176–81

[14] Kaup D J 1981 The solution of the general initial value problem for the full three-dimensional three-wave resonant interaction. *Physica* **3D** 374–95

[15] Kaup D J, Reiman A and Bers A 1979 Space-time evolution of nonlinear three-wave interactions I. Interaction in a homogenous medium. *Rev. Mod. Phys.* **51** 275–310

[16] Reiman A 1979 Space-time evolution of nonlinear three-wave interactions II. Interaction in a inhomogenous medium. *Rev. Mod. Phys.* **51** 311–30

[17] Tabor M and Gibbon J D 1986 Aspects of the Painlevé property for PDE's *Physica* **18D** 180–9

[18] Weiss J, Tabor M and Carnevale G 1983 The Painlevé property for PDE's *J. Math. Phys.* **24** 522–6

[19] Zakharov V E 1976 Exact solutions to the problem of the parametric interaction of three-dimensional wave packets. *Sov. Phys. Dokl.* **21** 322–3

[20] Zakharov V E and Manakov S V 1973 Resonant interaction of wave profiles in nonlinear media *Sov. Phys.–JETP Lett.* **18** 243–7

[21] Zakharov V E and Manakov S V 1975 The theory of resonance interaction of wave packets in nonlinear media *Sov. Phys.–JETP Lett.* **42** 842–50